Bounding the zeta function

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How big can the Riemann zeta function get?

ζ

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$$\zeta$$

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Extreme values of zeta

Conjecture (Farmer, Gonek, Hughes)

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = \exp\left(\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log T \log\log T}\right)$$

Bounds on extreme values of zeta

Theorem (Littlewood; Ramachandra and Sankaranarayanan, Soundararajan; Chandee and Soundararajan)

Under RH, there exists a C such that

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = O\left(\exp\left(C\frac{\log T}{\log\log T}\right)\right)$$

Theorem (Montgomery; Balasubramanian and Ramachandra; Balasubramanian; Soundararajan)

There exists a C' such that

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = \Omega\left(\exp\left(C'\sqrt{\frac{\log T}{\log\log T}}\right)\right)$$

The set-up:
Random Matrix Theory

Random matrix theory

- Nuclear physics (energy spectra of heavy nuclei).
- Quantum Chaos (is a system classically chaotic or integrable?)
- Significance of correlations in large data sets.
- Bus arrival times in Cuernavaca, Mexico & spacing between cars parked in London.
- And many, many other applications.
- Interesting and challenging mathematics.
- Models zeros of the Riemann zeta function.

Characteristic polynomials

Keating and Snaith modelled the Riemann zeta function with

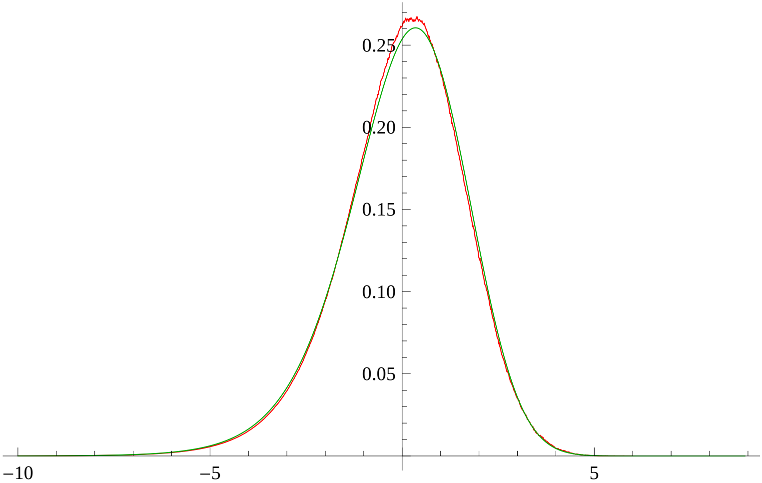
$$Z_{U_N}(\theta) := \det(I_N - U_N e^{-i\theta}) = \prod_{n=1}^{N} (1 - e^{i(\theta_n - \theta)})$$

where U_N is an $N \times N$ unitary matrix chosen with Haar measure.

The matrix size N is connected to the height up the critical line T via

$$N = \log \frac{T}{2\pi}$$

Characteristic polynomials



Graph of the value distribution of $\log|\zeta(\frac{1}{2}+\mathrm{i}t)|$ around the 10^{20}th zero (red), against the probability density of $\log|Z_{U_N}(0)|$ with $N = 42$ (green).

Characteristic polynomials: Normal distribution

Theorem (Selberg)

As $T \to \infty$,

$$\frac{1}{T} \operatorname{meas} \left\{ 0 \le t \le T \ : \ \frac{\log\left|\zeta(\frac{1}{2}+\mathrm{i}t)\right|}{\sqrt{\frac{1}{2}\log\log T}} \le C \right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{C} e^{-x^2/2} \, \mathrm{d}x$$

Theorem (Keating-Snaith)

As $N \to \infty$,

$$\mathbb{P}\left\{ \frac{\log\left|Z_{U_N}(0)\right|}{\sqrt{\frac{1}{2}\log N}} \le C \right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{C} e^{-x^2/2} \, \mathrm{d}x$$

Characteristic polynomials: Moments

Moment	Riemann Zeta Function	Characteristic polynomial

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$2k^{\text{th}}$	$\frac{G(k+1)^2}{G(2k+1)}a(k)(\log T)^{k^2}$	$\frac{G(k+1)^2}{G(2k+1)}N^{k^2}$

The Keating-Snaith conjecture

Theorem

$$\mathbb{E}\left[|Z_{U_N}(0)|^{2k}\right] \sim \frac{G^2(k+1)}{G(2k+1)} N^{k^2}$$

Conjecture

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+\mathrm{i}t)|^{2k}\,\mathrm{d}t \sim a(k)\frac{G^2(k+1)}{G(2k+1)}\left(\log\frac{T}{2\pi}\right)^{k^2}$$

where

$$a(k) = \prod_{\substack{p \\ \text{prime}}} \left(1-\frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\,\Gamma(k)}\right)^2 p^{-m}$$

Theorem (Gonek, Hughes, Keating)

A simplified form of our theorem is:

$$\zeta(\tfrac{1}{2} + \mathrm{i}t) = P(t; X) Z(t; X) + \textit{errors}$$

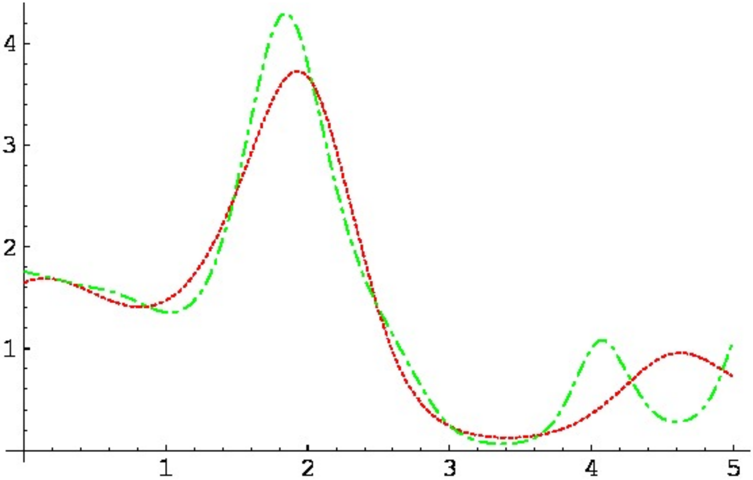
where

$$P(t; X) = \prod_{p \le X} \left(1 - \frac{1}{p^{\frac{1}{2} + \mathrm{i}t}}\right)^{-1}$$

and

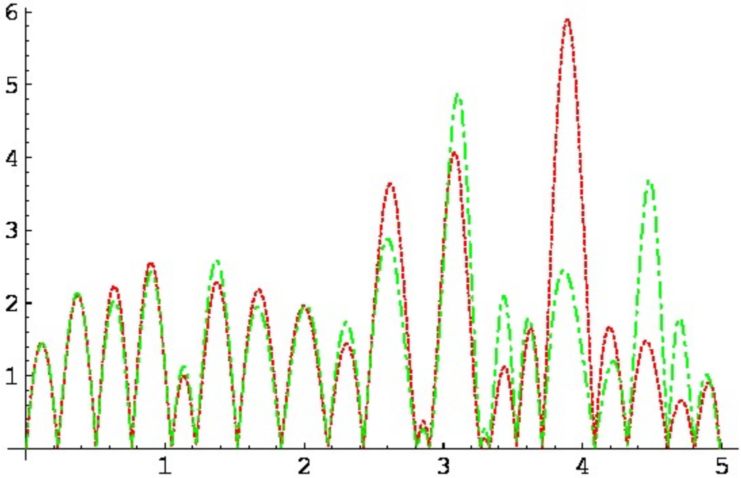
$$Z(t; X) = \exp\left(\sum_{\gamma_n} \mathrm{Ci}(|t - \gamma_n| \log X)\right)$$

An Euler-Hadamard hybrid: Primes only



Graph of $|P(t + t_0; X)|$, with $t_0 = \gamma_{10^{12}+40}$,
with $X = \log t_0 \approx 26$ (red) and $X = 1000$ (green).

An Euler-Hadamard hybrid: Zeros only



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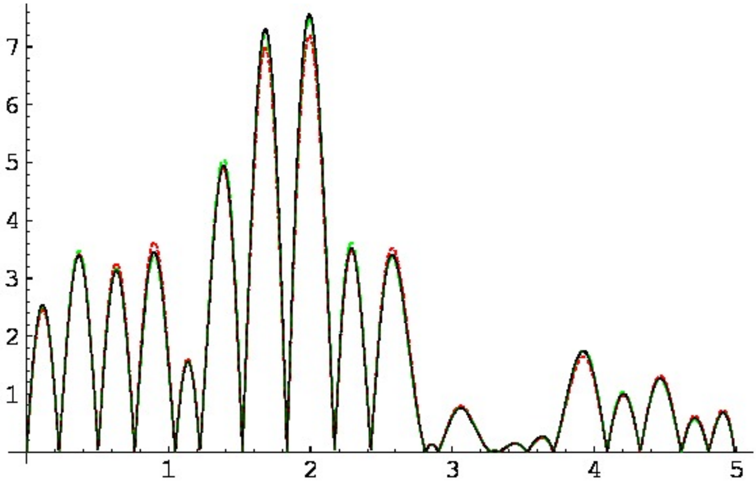
An Euler-Hadamard hybrid: Primes and zeros



Graph of $|\zeta(\frac{1}{2} + \mathrm{i}(t + t_0))|$ (black) and $|P(t + t_0; X)Z(t + t_0; X)|$, with $t_0 = \gamma_{10^{12}+40}$, with $X = \log t_0 \approx 26$ (red) and $X = 1000$ (green).

An Euler-Hadamard hybrid: Moments

Theorem

If $X = O(\log T)$

$$\frac{1}{T}\int_T^{2T} |P(t;X)|^{2k}\,\mathrm{d}t \sim a(k)(e^\gamma \log X)^{k^2}$$

Conjecture

If $X, T \to \infty$ *such that* $\frac{\log T}{\log X} \to \infty$

$$\frac{1}{T}\int_0^T |Z(t;X)|^{2k}\,\mathrm{d}t \sim \frac{G^2(k+1)}{G(2k+1)}\left(\frac{\log T}{e^\gamma \log X}\right)^{k^2}$$

This recovers the Keating-Snaith conjecture.

Argument 1:
Modeling Zeros With RMT

Extreme values of zeta: Zeros

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Model each block with the characteristic polynomial of an $N \times N$ random unitary matrix.
Find the smallest $K = K(M, N)$ such that choosing M independent characteristic polynomials of size N, almost certainly none of them will be bigger than K.

Extreme values of zeta: Zeros

Note that

$$\mathbb{P}\left\{\max_{1\le j\le M}\max_{\theta}|Z_{U_N^{(j)}}(\theta)|\le K\right\} = \mathbb{P}\left\{\max_{\theta}|Z_{U_N}(\theta)|\le K\right\}^M$$

Theorem

Let $0<\beta<2$*. If* $M=\exp(N^\beta)$*, and if*

$$K=\exp\left(\sqrt{\left(1-\tfrac{1}{2}\beta+\varepsilon\right)\log M\log N}\right)$$

then

$$\mathbb{P}\left\{\max_{1\le j\le M}\max_{\theta}|Z_{U_N^{(j)}}(\theta)|\le K\right\}\to 1$$

as $N\to\infty$ *for all* $\varepsilon>0$*, but for no* $\varepsilon<0$*.*

Extreme values of zeta: Zeros

Recall

$$\zeta(\tfrac{1}{2} + \mathrm{i}t) = P(t; X)Z(t; X) + \text{errors}$$

and that $Z(t; X)$ can be modelled by characteristic polynomials of size

$$N = \frac{\log T}{e^{\gamma} \log X}$$

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and that $Z(t; X)$ can be modelled by characteristic polynomials of size

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Therefore the previous theorem suggests

Conjecture

If $X = \log T$*, then*

$$\max_{t\in[0,T]} |Z(t; X)| = \exp\left((\frac{1}{\sqrt{2}} + o(1)) \sqrt{\log T \log\log T} \right)$$

Extreme values of zeta: Zeros

Theorem

By the PNT, if $X = \log T$ *then for any* $t \in [0, T]$*,*

$$P(t; X) = O\left(\exp\left(C\frac{\sqrt{\log T}}{\log\log T}\right)\right)$$

Thus one is led to the max values conjecture

Conjecture

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2} + \mathrm{i}t)| = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log T \log\log T}\right)$$

Argument 2:
Random Prime Model

Extreme values of zeta: Primes

First note that

$$P(t;X) = \exp\left(\sum_{p\leq X} \frac{1}{p^{1/2+it}}\right) \times O(\log X)$$

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Treat $p^{-\mathrm{i}t}$ as independent random variables distributed uniformly on the unit circle.
This suggests the distribution of

$$\Re\mathfrak{e} \sum_{p\le X} \frac{p^{-\mathrm{i}t}}{\sqrt{p}}$$

tends to Gaussian with mean 0 and variance $\frac{1}{2}\log\log X$ as $X \to \infty$.

Extreme values of zeta: Primes

We let $X = \exp(\sqrt{\log T})$ and model the maximum of $P(t;X)$ by finding the maximum of the Gaussian random variable sampled $T(\log T)^{1/2}$ times. This suggests

$$\max_{t\in[0,T]} |P(t;X)| = O\left(\exp\left((\frac{1}{\sqrt{2}}+\varepsilon)\sqrt{\log T \log\log T}\right)\right)$$

for all $\varepsilon > 0$ and no $\varepsilon < 0$.

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for all $\varepsilon > 0$ and no $\varepsilon < 0$.
For such a large X, random matrix theory suggests that

$$\max_{t\in[0,T]} |Z(t;X)| = O\left(\exp\left(\sqrt{\log T}\right)\right).$$

This gives another justification of the large values conjecture.

Argument 3:
Zeros and Primes

Extreme values of zeta: Zeros and primes

If $X = \exp(\log^{\alpha} T)$ with $0 < \alpha < \frac{1}{2}$, then the largest values of $Z(t; X)$ and $P(t; X)$ are of approximately the same size and both will contribute to the largest values of $\zeta(\frac{1}{2} + \mathrm{i}t)$.

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Specifically, $|Z(t;X)|$ gets as large as

$$\exp\left(\frac{1}{\sqrt{2}}\sqrt{(1-2\alpha)\log T \log\log T}\right),$$

and $|P(t;X)|$ gets as large as

$$\exp\left(\sqrt{\alpha \log T \log\log T}\right).$$

The product of these two is greater than our conjecture for all α between 0 and $1/2$.

Extreme values of zeta: Zeros and primes

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Since $\zeta(\frac{1}{2}+\mathrm{i}t)$ is close to its maximum value over a window of size $C/\log T$, we wish to find the smallest K such that

$$\operatorname{meas}\{0 < t < T \ : \ |P(t;X)||Z(t;X)| \geq K\} \ll \frac{1}{\log T},$$

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A saddle point approximation argument yields that if $X = \exp(\log^\alpha T)$ then for all $0 < \alpha < 1/2$,

$$K = \exp\left(\left(\sqrt{\frac{1}{2}} + o(1)\right)\sqrt{\log T \log\log T}\right)$$

Argument 4:
Moments

Extreme values of zeta: Moments

Let

$$m_T = \max_{0 \le t \le T} |\zeta(\tfrac{1}{2} + \mathrm{i}t)|$$

and

$$I_k = \frac{1}{T} \int_0^T |\zeta(\tfrac{1}{2} + \mathrm{i}t)|^{2k} \, \mathrm{d}t$$

Then trivially, $(m_T)^{2k} \geq I_k$.

Extreme values of zeta: Moments

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Then trivially, $(m_T)^{2k} \ge I_k$.
In the other direction, if t_0 is the height of the maximum value, then

$$(m_T)^{2\ell} \ll 2^{2\ell} \log T \int_{t_0 - C/\log T}^{t_0 + C/\log T} |\zeta(\tfrac{1}{2} + \mathrm{i}t)|^{2\ell} \, \mathrm{d}t \ll 2^{2\ell} I_\ell T \log T$$

Hence,

$$(I_k)^{1/2k} \le m_T \ll (T \log T)^{1/2\ell} (I_\ell)^{1/2\ell}$$

Extreme values of zeta: Moments

Using known large-k asymptotics for the constant in the Keating-Snaith moment conjecture (which is only made for finite k), one can show that it cannot hold for

$$k \geq \sqrt{\frac{8 \log T}{\log \log T}}$$

If it does hold for such a large value of k, then

$$\max_{t \in [0,T]} |\zeta(\tfrac{1}{2} + \mathrm{i}t)| = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1) \right) \sqrt{\log T \log \log T} \right)$$

Related Conjectures

The extreme values of $\left|\zeta(\frac{1}{2}+\mathrm{i}t)\right|$ were deduced from knowledge of $\mathfrak{Re}\log\zeta(\frac{1}{2}+\mathrm{i}t)$.
Similar arguments work for $\mathfrak{Im}\zeta(\frac{1}{2}+\mathrm{i}t)$, and hence for $S(t)$, the error term in the number of zeros of the zeta-function up to height t.

Conjecture

$$\limsup_{t\to\infty} \frac{S(t)}{\sqrt{\log t \log\log t}} = \frac{1}{\pi\sqrt{2}}.$$

Extreme values: Other L-functions

For the Symplectic family of real primitive Dirichlet L-functions $L(s, \chi_d)$, where $\chi_d = \left(\frac{d}{\cdot}\right)$)

Conjecture

$$\max_{|d| \leq D} \left| L(\tfrac{1}{2}, \chi_d) \right| = O\left(\exp\left((1+\varepsilon) \sqrt{\log D \log\log D} \right) \right)$$

for all $\varepsilon > 0$ *and for no* $\varepsilon < 0$.

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for all $\varepsilon > 0$ *and for no* $\varepsilon < 0$.

For the Orthogonal family of Dirichlet series associated to holomorphic cusp forms of weight k and level N:

Conjecture

$$\max_{\substack{f \in S_k(\Gamma_0(N)) \\ kN \le D}} \left| L(\tfrac{k}{2}, f) \right| = O\left(\exp\left((1+\varepsilon) \sqrt{\log D \log\log D} \right) \right)$$

for all $\varepsilon > 0$ *and for no* $\varepsilon < 0$.

Summary

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We gave several arguments (based on random matrix theory, a random prime model, and moments) supporting the conjecture that

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right) \sqrt{\log T \log\log T} \right)$$